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Remarks on the exotic central extension of the planar Galilei group

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Abstract

Some issues in relating the central extensions of the planar Galilei group to parameters in the corresponding relativistic theory are discussed.

The fact that the Galilei group in (2+1) dimensions admits a two-parameter central extension has been known for a long time [1]. The first of these extensions is identified as the mass. Two years ago, we pointed out that the second, more exotic, extension is related to the spin of the particle [2]. This has recently been the subject of some discussion. In [3], it is argued on the basis of a nonrelativistic model that one can have spin with no extension of the algebra. The paper by Duval and Horvathy presents an analysis of the nonrelativistic limit and the extensions obtained by using a regularization [4]. While we agree with the analysis of [4], we feel that the following remarks offer further clarification.

The Cartan-Poincaré form for single particle Galilean dynamics is given by

$$\omega = dp^i dx^i + \frac{\kappa}{2m^2}\epsilon^{ij} dp^i dp^j - \frac{p^i dp^i dt}{m} \tag{1}$$

where m and κ are the extension parameters. This form and many of the related questions have been extensively studied [5]. The symplectic form for a free relativistic particle can be written as [6]

$$\Omega = -dp_a dx^a + \frac{S}{2}\frac{\epsilon^{abc} p_a dp_b dp_c}{(p^2)^{\frac{3}{2}}} \tag{2}$$

Our basic observation was that a nonrelativistic expansion of the latter using $p_0 \approx mc + (p^i p^i / 2mc)$ will lead to (1) with $\kappa = S/c^2$.

Note that one may think of the nonrelativistic theory in two ways, either as $p^i p^i \ll m^2 c^2$ in which case it is a nonrelativistic approximation, or as $c \to \infty$ in which case it is a nonrelativistic limit. We shall consider the first way, the approximation $p^i p^i \ll m^2 c^2$. Recall that the second extension may also be characterized by the nonzero commutator of the Galilei boost generators K_i. With $p^i p^i \ll m^2 c^2$, the Poisson bracket of the Lorentz boost generators J_i associated with (2) becomes

$$[K_i, K_j] \approx \epsilon_{ij}(S/c^2) \tag{3}$$

where $J^i \approx c\epsilon^{ij} K_j$ and the right hand side arises from the nonrelativistic approximation to J_0. Thus the second extension κ of the Galilei group is S/c^2. Numerically, the appropriate dimensionless quantity Suv/c^2, u, v being typical velocities, may be small in the nonrelativistic approximation. Nevertheless, S/c^2 remains as an extension parameter characterizing the algebra of the Galilei boost generators; mathematically, we do obtain the extended Galilei group. If one takes $c \to \infty$, then we need $S \to \infty$ to get finite κ. One may feel uncomfortable with this, but it is also mathematically correct.

We now look at the the Levy-Léblond equations for a spin-half particle given by [7]

$$i\frac{\partial \phi}{\partial t} + p_{-}\chi \;\; = \;\; 0$$

$$p_+\phi + 2m\chi \ = \ 0 \tag{4}$$

where $p_\pm = p_1 \pm ip_2$. In the free case, this model is equivalent to the Lagrange density

$$\mathcal{L} = \phi^* i\frac{\partial\phi}{\partial t} \ - \ \phi^*\frac{p_-p_+}{2m}\phi \tag{5}$$

By assigining a new transformation law to the field ϕ as $\phi \to \phi' = \exp(i\frac{\theta}{2} + i\alpha\theta)\phi$ under rotations, the angular momentum is seen to be shifted as $L_0 \to L_0 + \alpha\int\phi^*\phi$. Since the number operator commutes with the Hamiltonian and various other operators of interest, the algebra is unchanged. Thus, in discussing nonrelativistic models, it is important to keep in mind that, at the level of the Galilei group, there is nothing to fix the zero of angular momentum. One can always add a constant without changing the algebra. The ambiguity of identification of spin can never be resolved within the Galilei symmetry. So the whole idea of relating spin to the second extension does not work if one is limited to the Galilean group. But this is not our point, rather we note that there is a definition of spin at the relativistic level and this can be interpreted in terms of the second extension of the Galilean group.

The situation is similar with the case of the mass. At the level of the Galilei group, the mass is a free parameter. It is defined by

$$H = \frac{p^2}{2m} \tag{6}$$

(or by the commutation rule $[K_i, P_j]$ which is related to the above.) There is also an additive constant we can put in H, writing

$$H = Mc^2 + \frac{p^2}{2m} \tag{7}$$

At the Galilean level, there is no reason why M and m should be the same; the rest mass (or rest energy) and the inertia are independent parameters. Relativistically, there is the definition of mass from the Poincaré group as the value of $p^\mu p_\mu$. If we want the Hamiltonian H to be the limit of this, then $M = m$. Thus relativistic considerations give an interpretation to the inertial mass m (the first extension of the Galilei group) as the rest mass M. What we are doing is to link similarly the relativistic definition of spin to the second extension. (In [4], this freedom in the zero of energy and spin is used to 'renormalize' the parameters in the Galilean limit, explicitly carried out by modification of the symplectic forms by including some 'trivial' extensions of the Poincaré group.)

The similarity between mass as rest energy and the spin can be seen even more clearly by starting from a Galilean theory which has a second extension. As noted in [3], we can achieve this trivially by shifting the unextended Galilei boost generators

$$K_i \to \tilde{K}_i = K_i + \frac{\kappa}{2m}\epsilon_{ij}p_j \tag{8}$$

This $\tilde{K}_i$ shows the second extension in its Poisson bracket with itself. However, this modification changes the transformation law of x_i. We now have

$$[v \cdot \tilde{K}, x_i] = v_i t - \frac{\kappa}{2m} v_j \epsilon_{ji} \tag{9}$$

We no longer have the expected $x \to x + vt$. Define

$$q_i = x_i + \frac{\kappa}{2m^2} \epsilon_{ji} p_j \tag{10}$$

It is easily checked that $[v \cdot \tilde{K}, q_i] = v_i t$. Thus q_i is the correctly transforming coordinate, but now $[q_i, q_j] \neq 0$. The angular momentum L_0 must be defined by

$$\begin{aligned} [L_0, q_i] &= \epsilon_{ij} q_j \\ [L_0, p_i] &= \epsilon_{ij} p_j \end{aligned} \tag{11}$$

This is easily solved to obtain

$$L_0 = \alpha + \epsilon_{ij} q_i p_j - \frac{\kappa}{2m^2}\, p^2 \tag{12}$$

where there is the freedom of an additive constant α. Note the similarity of this equation, with the two parameters α and κ, to equation (7) for the Hamiltonian, with the parameters M and m. Equations (8)-(12) define a Galilean theory with the second extension κ, and so far, the parameters α and κ are not related. Now we can ask the question: can we choose the parameters α and κ (just as we chose M in terms of m) such that equation (12) is the low momentum limit of the relativistic expression? The answer is that it can be done if $\alpha = \kappa c^2$. But α is the spin from the relativistic point of view and so we have the result of [2]. In this way of interpreting the result, we are not taking $c \to \infty$, but rather asking whether the *a priori* unrelated parameters of the nonrealtivistic theory with the extensions can be related if viewed as the low momentum limit of a relativistic theory.

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References

[1] J.-M. Lévy-Leblond, in E. Loebl, *Group Theory and Applications*, Academic Press, New York (1972); D.R. Grigore, J. Math. Phys., **37** (1996) 460; S.K. Bose, Commun. Math. Phys. bf 169 (1995) 385.

[2] R. Jackiw and V.P. Nair, Phys. Lett. **B480** (2000) 237.

[3] C.R. Hagen, Phys. Lett. **B539** (2002) 168.

[4] C. Duval and P.A. Horvathy, hep-th/0209166.

[5] J. Lukierski, P.C. Stichel and W.J. Zakrzewski, Ann. Phys. (NY) **260** (1997) 224; C. Duval and P.A. Horvathy, Phys. Lett. **B479** (2000) 284.

[6] R. Jackiw and V.P. Nair, Phys. Rev. **D43** (1991) 1933; see also B. Skagerstam and A. Stern, Int. J. Mod. Phys. **A5** (1990) 1575; M.S. Plyushchay, Phys. Lett. **B248** (1990) 107; Nucl. Phys. **B362** (1991), 54; A. Ballesteros, M. Gadella and M. del Olmo, J. Math. Phys. **33** (1992) 3379; S. Ghosh, Phys. Lett. **B338** (1994) 235.

[7] J.-M. Lévy-Leblond, Comm. Math. Phys. **6** (1967) 286.